Concise Derivation of Extensive Coordinate Conversion Formulae in the Gauss-Krüger Projection

Kazushige KAWASE

Abstract

Although the set of Gauss-Krüger projection formulae recognized and used worldwide thus far is "the 2nd formulae", which is limited to use within certain longitudinal zones, the recently discovered usefulness of "the 1st formulae", which can be applied to extensive areas on the globe, has been understood from a new perspective. The formulae are favorable in view of the modern computer environment, in which they are of importance for recognizing details comprehensively, not only in the discipline of survey and mapping but also in geospatial information management. In this paper, I try to elaborate a comprehensive derivation of these formulae, including those for meridian convergence and scale factor, so that anyone interested in geospatial information management can understand them with minimum background knowledge.

1. Introduction

In 1912, just one century ago, Johannes Heinrich Louis Krüger published "*Konforme Abbildung des Erdellipsoids in der Ebene*" (Conformal Mapping of the Earth Ellipsoid to the Plane; Krüger, 1912). In this article, two kinds of formulae to describe coordinate conversion appeared. The set of formulae that appeared in the first part of the article (which we will hereafter call *"the 1st formulae"*) is applicable to conversion for extensive areas on the globe. In contrast, the set of formulae that appeared in the second part of the article (which we will hereafter call *"the 2nd formulae"*) is restricted to usage within certain longitudinal zones. Notwithstanding their capacity for general usage, *the 1st formulae* appear to have been less useful than *the 2nd formulae*, owing to the intricacies of the whole formulaic system as well as the roundabout derivation process described in Krüger's original article.

Under these circumstances, Karney (Karney, 2011) opened our eyes to the fact that *the 1st formulae* are in harmony with the present computing environment. Based on this fact, some commentaries (e.g., Deakin et al., 2011) have been reported. Deakin et al. explain these formulae both via Gauss-Lambert and Gauss-Schreiber projections. Although these commentaries help us to grasp the concepts underlying *the 1st formulae*, there is room for improvement in terms of reducing the amount of indispensable background knowledge needed for comprehension.

In this short report, a comprehensive derivation process for the extensive coordination conversion formulae in the Gauss-Krüger projection is explicated using nothing but a basic concept of elementary complex analysis as background knowledge. In this paper, I first show consecutive step action flow to describe how to perform the necessary steps in order to achieve the desired conversion (section 2). Next, in section 3, I present all the formulae, including those for *meridian convergence* γ and *scale factor* m. In section 4, I give a step by step explanation of their derivation; finally, I conclude this report in section 5.

2. Displaying step action flow for conversion

The consecutive step action flow of the extensively applicable coordinate conversion in the Gauss-Krüger projection is presented as follows:

1. Regard the earth as an oblate ellipsoid with *semi-major axis* a and the *3rd flattening* n. (As a rule, the earth ellipsoid is identified by two parameters—the semi-major axis and the *inverse 1st flattening* F, hereafter however, we use $n=1/(2F-1)$ instead of F.)
   Assign *geographic latitude* φ and *longitude* λ as

geographic coordinates on the surface of the earth ellipsoid.

2. Transform geographic coordinates to Cartesian coordinates by "normal" Mercator projection. At this point, introduce *isometric latitude* ψ, which is defined as

$$\psi = \mathrm{gd}^{-1}\varphi - \frac{2\sqrt{n}}{1+n}\tanh^{-1}\left(\frac{2\sqrt{n}}{1+n}\sin\varphi\right). \tag{1}$$

Here, $\mathrm{gd}^{-1}x$ denotes the inverse function of the Gudermannian function gd x, which is defined as

$$\begin{aligned}\mathrm{gd}\,x &= \int_0^x \frac{\mathrm{d}t}{\cosh t} \\ &= \sin^{-1}\tanh x = \tan^{-1}\sinh x.\end{aligned} \tag{2}$$

3. Assign a desired meridian, with longitude λ_0, as a central meridian and set $\Delta\lambda = \lambda - \lambda_0$. Regard the acquired Mercator projection plane as a complex plane ϑ in which the central meridian is set up as the real axis, and the equator, as the imaginary axis; i.e., $\vartheta = \psi + \mathrm{i}\Delta\lambda$, where $\mathrm{i} = \sqrt{-1}$ denotes the imaginary unit.

4. Perform a conformal mapping from the ϑ-plane into a complex plane $\zeta' = \xi' + \mathrm{i}\eta'$ using the Gudermannian function, which, as a matter of course, can be regarded as a holomorphic function (complex regular function) on the domain under discussion; i.e.,

$$\zeta' = \mathrm{gd}\,\vartheta = \mathrm{gd}(\psi + \mathrm{i}\Delta\lambda). \tag{3}$$

This mapping is performed in order to normalize the range of the variable of isometric latitude ($-\infty \le \psi \le \infty$) to that of *conformal latitude* ($-\pi/2 \le \chi \le \pi/2$) on the real axis of the ζ'-plane in order to enable the dual-form transformations performed in the next step, i.e.,

$$\mathrm{gd}\,\psi = \sin^{-1}\tanh\psi = \tan^{-1}\sinh\psi = \chi. \tag{4}$$

5. Perform a conformal mapping from the ζ'-plane into a complex plane $\zeta = \xi + \mathrm{i}\eta$ using a holomorphic function of the form

$$\zeta = \zeta' + \sum_{k=1}^{\infty}\alpha_k \sin 2k\zeta', \tag{5}$$

and inverse form,

$$\zeta' = \zeta - \sum_{k=1}^{\infty}\beta_k \sin 2k\zeta. \tag{6}$$

At this point, determine the values of the Fourier sine coefficients $\{\alpha_k\}$ that appear in formula (5) so that the value of argument χ on the real axis of the ζ'-plane coincides with *rectifying latitude* μ, defined as scaled latitude by meridian arc length, i.e.,

$$\mu = \chi + \sum_{k=1}^{\infty}\alpha_k \sin 2k\chi. \tag{7}$$

6. Finally, multiply both coordinates ξ and η by the product of *central meridian scale factor* m_0 and *rectifying radius* A in order to obtain plane rectangular coordinates *easting* X and *northing* Y, which coincide with an appropriate scale of the real world.
The rectifying radius is defined as the corresponding radius of the circle whose perimeter coincides with the entire meridian arc length of the earth ellipsoid.

We note that with regard to the details of the determination of the Fourier sine coefficients $\{\alpha_k\}$ and $\{\beta_k\}$ that appear in step 5, as well as the Fourier sine coefficients $\{\delta_k\}$ that appear in the first part of formula (13), presented in section 3, a comprehensive summary has already been given by Kawase (Kawase, 2011).

3. Displaying all the extensive coordinate conversion formulae in the Gauss-Krüger projection

On the basis of the step action flow shown in the previous section, we write all the forward and inverse coordinate conversion formulae in the Gauss-Krüger projection. Note that the roles of coordinates X and Y in the Japanese surveying and mapping community is contrary to that displayed below, i.e., in Japan, X

traditionally denotes northing while Y denotes easting.

(φ, λ_0, $\Delta\lambda=\lambda-\lambda_0$: given)

$$X = m_0 A\left(\eta' + \sum_{k=1}^{\infty}\alpha_k \cos 2k\xi' \sinh 2k\eta'\right), \quad Y = m_0 A\left(\xi' + \sum_{k=1}^{\infty}\alpha_k \sin 2k\xi' \cosh 2k\eta'\right) \tag{8}$$

$$\gamma = \tan^{-1}\left(\frac{\tau\sqrt{1+\tan^2\chi} + \sigma\tan\chi\tan\Delta\lambda}{\sigma\sqrt{1+\tan^2\chi} - \tau\tan\chi\tan\Delta\lambda}\right), \quad m = m_0\frac{A}{a}\sqrt{\left\{1+\left(\frac{1-n}{1+n}\tan\varphi\right)^2\right\}\frac{\sigma^2+\tau^2}{\tan^2\chi+\cos^2\Delta\lambda}} \tag{9}$$

$$\tan\chi = \sinh\left(\tanh^{-1}\sin\varphi - \frac{2\sqrt{n}}{1+n}\tanh^{-1}\left(\frac{2\sqrt{n}}{1+n}\sin\varphi\right)\right) \tag{10}$$

$$\xi' = \tan^{-1}\left(\frac{\tan\chi}{\cos\Delta\lambda}\right), \quad \eta' = \tanh^{-1}\left(\frac{\sin\Delta\lambda}{\sqrt{1+\tan^2\chi}}\right) \tag{11}$$

$$\sigma = 1 + \sum_{k=1}^{\infty} 2k\alpha_k \cos 2k\xi' \cosh 2k\eta', \quad \tau = \sum_{k=1}^{\infty} 2k\alpha_k \sin 2k\xi' \sinh 2k\eta' \tag{12}$$

(λ_0, X, Y: given)

$$\varphi = \chi + \sum_{k=1}^{\infty}\delta_k \sin 2k\chi, \quad \Delta\lambda = \tan^{-1}\left(\frac{\sinh\eta'}{\cos\xi'}\right), \quad \lambda = \lambda_0 + \Delta\lambda \tag{13}$$

$$\gamma = \tan^{-1}\left(\frac{\tau' + \sigma'\tan\xi'\tanh\eta'}{\sigma' - \tau'\tan\xi'\tanh\eta'}\right), \quad m = m_0\frac{A}{a}\sqrt{\left\{1+\left(\frac{1-n}{1+n}\tan\varphi\right)^2\right\}\frac{\cos^2\xi'+\sinh^2\eta'}{\sigma'^2+\tau'^2}} \tag{14}$$

$$\xi = \frac{1}{m_0 A}Y, \quad \eta = \frac{1}{m_0 A}X \tag{15}$$

$$\xi' = \xi - \sum_{k=1}^{\infty}\beta_k \sin 2k\xi \cosh 2k\eta, \quad \eta' = \eta - \sum_{k=1}^{\infty}\beta_k \cos 2k\xi \sinh 2k\eta \tag{16}$$

$$\sigma' = 1 - \sum_{k=1}^{\infty} 2k\beta_k \cos 2k\xi \cosh 2k\eta, \quad \tau' = \sum_{k=1}^{\infty} 2k\beta_k \sin 2k\xi \sinh 2k\eta \tag{17}$$

$$\chi = \sin^{-1}\left(\frac{\sin\xi'}{\cosh\eta'}\right) \tag{18}$$

4. Explanation of the derivation of each formula

4.1 Derivation of the main conversion

In this section, we confirm the formulae displayed in the previous section one by one. First of all, we define real variables σ, τ, and σ', τ' as

$$\sigma - \mathrm{i}\tau = \frac{\mathrm{d}\zeta}{\mathrm{d}\zeta'}, \quad \sigma' + \mathrm{i}\tau' = \frac{\mathrm{d}\zeta'}{\mathrm{d}\zeta} = \frac{1}{\sigma - \mathrm{i}\tau}. \tag{19}$$

From formulae (5), (6), and (19), along with the properties of complex trigonometric functions in terms of the real and imaginary parts of their arguments (e.g., Abramowitz et al., 1964) as well as the procedure for step 6 in section 2, it is not hard to derive formulae (8), (12), (15), (16), and (17).

As for formula (10)*, we can easily derive it from formulae (1) and (4) directly.

4.2 Derivation of the relation between variables (χ, $\Delta\lambda$) and (ξ', η')

Next, we confirm the relation between variables (χ, $\Delta\lambda$) and (ξ', η'). Note that the results that we shall now show below may also be derived from other citations (e.g., Weisstein, 2008).

We start from the relation

$$\zeta' = \mathrm{gd}\,\vartheta = \mathrm{gd}\left(\mathrm{gd}^{-1}\chi + \mathrm{i}\Delta\lambda\right), \tag{20}$$

which is derived from formulae (3) and (4) and yields

$$\mathrm{gd}^{-1}\zeta' = \mathrm{gd}^{-1}\chi + \mathrm{i}\Delta\lambda. \tag{21}$$

By operating the hyperbolic tangent on both sides of formula (21), we get

$$\sin\zeta' = \tanh\left(\mathrm{gd}^{-1}\chi + \mathrm{i}\Delta\lambda\right). \tag{22}$$

Then, applying the property of complex sine functions in terms of the real and imaginary parts of their arguments to the left hand side of formula (22) and addition formulae for complex hyperbolic functions to the right hand side of formula (22), we obtain

$$\begin{aligned} &\sin\xi'\cosh\eta' + \mathrm{i}\cos\xi'\sinh\eta' \\ &= \frac{\sin\chi + \tanh(\mathrm{i}\Delta\lambda)}{1 + \sin\chi\tanh(\mathrm{i}\Delta\lambda)} = \frac{\sin\chi + \mathrm{i}\tan\Delta\lambda}{1 + \mathrm{i}\sin\chi\tan\Delta\lambda} \\ &= \frac{\left(1+\tan^2\Delta\lambda\right)\sin\chi + \mathrm{i}\left(1-\sin^2\chi\right)\tan\Delta\lambda}{1+\sin^2\chi\tan^2\Delta\lambda}. \end{aligned} \tag{23}$$

Regarding the equivalence of the real and imaginary parts of both sides of formula (23) as simultaneous equations, and bearing $|\sin\chi| \le 1$ in mind†, we can obtain the appropriate solutions with respect to $\sin\chi$ and $\tan\Delta\lambda$ as

$$\sin\chi = \frac{\sin\xi'}{\cosh\eta'}, \quad \tan\Delta\lambda = \frac{\sinh\eta'}{\cos\xi'}, \tag{24}$$

from which we derive the second part of formula (13) and formula (18). Using the results of formula (24), we can also obtain other expressions of relation as

$$\cos\chi = \frac{\sqrt{\cos^2\xi' + \sinh^2\eta'}}{\cosh\eta'}, \tag{25}$$

$$\tan\chi = \frac{\sin\xi'}{\sqrt{\cos^2\xi' + \sinh^2\eta'}}, \tag{26}$$

$$\cos\Delta\lambda = \frac{\cos\xi'}{\sqrt{\cos^2\xi' + \sinh^2\eta'}}, \tag{27}$$

$$\sin\Delta\lambda = \frac{\sinh\eta'}{\sqrt{\cos^2\xi' + \sinh^2\eta'}}, \tag{28}$$

which yield formula (11) when formulae (26) and (27) for ξ' and formulae (25) and (28) for η', are combined.

4.3 Derivation of meridian convergence

With regard to meridian convergence, we start from its definition. Bearing the well-known characteristic of derivation of holomorphic function in mind, we can see that

* An improved expression is presented by Karney (Karney, 2011).

† By this restriction, the other apparent set of solutions $\sin\chi = \cosh\eta'/\sin\xi'$, $\tan\Delta\lambda = \cos\xi'/\sinh\eta'$ is discarded.

$$\tan\gamma \equiv \left.\frac{\mathrm{d}Y}{\mathrm{d}X}\right|_{\varphi=\text{const}} = \frac{\dfrac{\partial Y}{\partial\Delta\lambda}}{\dfrac{\partial X}{\partial\Delta\lambda}} = \frac{\dfrac{\partial\xi}{\partial\Delta\lambda}}{\dfrac{\partial\eta}{\partial\Delta\lambda}} = \frac{-\mathrm{Im}\left(\dfrac{\mathrm{d}\zeta}{\mathrm{d}(\psi+\mathrm{i}\Delta\lambda)}\right)}{\mathrm{Re}\left(\dfrac{\mathrm{d}\zeta}{\mathrm{d}(\psi+\mathrm{i}\Delta\lambda)}\right)}. \tag{29}$$

Here, we have referenced two facts. The first is that the composite function of holomorphic functions must also be a holomorphic function, and the other is that Cauchy-Riemann equations must be satisfied with respect to arbitrary holomorphic functions.

With respect to the complex derivative that appeared in the fraction of the final result for formula (29), it follows from formulae (2), (3), and (19) that we can calculate it explicitly as

$$\begin{aligned}\frac{\mathrm{d}\zeta}{\mathrm{d}(\psi+\mathrm{i}\Delta\lambda)} &= \frac{\mathrm{d}\zeta}{\mathrm{d}\zeta'}\cdot\frac{\mathrm{d}\zeta'}{\mathrm{d}(\psi+\mathrm{i}\Delta\lambda)} = (\sigma-\mathrm{i}\,\tau)\frac{\mathrm{d}\,\mathrm{gd}(\psi+\mathrm{i}\Delta\lambda)}{\mathrm{d}(\psi+\mathrm{i}\Delta\lambda)} = \frac{\sigma-\mathrm{i}\,\tau}{\cosh(\psi+\mathrm{i}\Delta\lambda)}\\ &= \frac{\sigma-\mathrm{i}\,\tau}{\cosh\psi\cos\Delta\lambda+\mathrm{i}\sinh\psi\sin\Delta\lambda} = \frac{1}{\cosh\psi\cos\Delta\lambda}\cdot\frac{\sigma-\mathrm{i}\,\tau}{1+\mathrm{i}\tanh\psi\tan\Delta\lambda}\\ &= \frac{1}{\cosh\psi\cos\Delta\lambda}\cdot\frac{(\sigma-\tau\tanh\psi\tan\Delta\lambda)-\mathrm{i}(\tau+\sigma\tanh\psi\tan\Delta\lambda)}{1+\tanh^2\psi\tan^2\Delta\lambda}.\end{aligned} \tag{30}$$

In this case, the essential part which formula (30) implies is merely the numerator in the second fraction of the final result since all we have to do is to take the ratio of the imaginary part to the real part of the result of formula (30) in accordance with the final result of formula (29).

Bearing the above and formula (4) in mind, it is not hard to see that

$$\tan\gamma = \frac{\tau+\sigma\tanh\psi\tan\Delta\lambda}{\sigma-\tau\tanh\psi\tan\Delta\lambda} = \frac{\tau+\sigma\sin\chi\tan\Delta\lambda}{\sigma-\tau\sin\chi\tan\Delta\lambda}, \tag{31}$$

and hence

$$\begin{aligned}\gamma &= \tan^{-1}\left(\frac{\tau+\sigma\sin\chi\tan\Delta\lambda}{\sigma-\tau\sin\chi\tan\Delta\lambda}\right)\\ &= \tan^{-1}\left(\frac{\tau\sqrt{1+\tan^2\chi}+\sigma\tan\chi\tan\Delta\lambda}{\sigma\sqrt{1+\tan^2\chi}-\tau\tan\chi\tan\Delta\lambda}\right),\end{aligned} \tag{32}$$

which corresponds to the first result of formula (9). We can also obtain the first part of formula (14) from the above result along with formulae (19) and (24).

4.4 Derivation of scale factor

With regard to scale factor, we also start from its definition as

$$\begin{aligned}m^2 &\equiv \frac{m_0^2}{N_\varphi^2\cos^2\varphi}\cdot\frac{(\mathrm{d}X)^2+(\mathrm{d}Y)^2}{(\mathrm{d}\psi)^2+(\mathrm{d}\Delta\lambda)^2}\\ &= \frac{m_0^2}{N_\varphi^2\cos^2\varphi}\left|\frac{\mathrm{d}(Y+\mathrm{i}X)}{\mathrm{d}(\psi+\mathrm{i}\Delta\lambda)}\right|^2.\end{aligned} \tag{33}$$

Here, N_φ denotes the radius of curvature in the prime vertical for the earth ellipsoid, which is a function of φ. Bearing in mind the following relations as well as formula (4) and the interim results of formula (30), it is not hard to see that which yields the last part of formula (9). We can also obtain the last part of formula (14) from the above result and formulae (19), (26), and (27).

$$\begin{aligned}\frac{1}{N_\varphi^2\cos^2\varphi} &= \frac{1}{a^2}\left\{1-\frac{4n}{(1+n)^2}\sin^2\varphi\right\}\cdot\frac{1}{\cos^2\varphi}\\ &= \frac{1}{a^2}\left\{1+\left(\frac{1-n}{1+n}\tan\varphi\right)^2\right\},\end{aligned} \tag{34}$$

$$\frac{\mathrm{d}(Y+\mathrm{i}X)}{\mathrm{d}(\psi+\mathrm{i}\Delta\lambda)} = A\frac{\mathrm{d}(\xi+\mathrm{i}\eta)}{\mathrm{d}(\psi+\mathrm{i}\Delta\lambda)} = A\frac{\mathrm{d}\zeta}{\mathrm{d}(\psi+\mathrm{i}\Delta\lambda)} \tag{35}$$

Thus, we finally achieve overall comprehension of the extensive coordination conversion formulae in the Gauss-Krüger projection.

$$
\begin{aligned}
m^2 &= \left(\frac{m_0 A}{a}\right)^2 \left\{1+\left(\frac{1-n}{1+n}\tan\varphi\right)^2\right\} \left|\frac{\mathrm{d}\zeta}{\mathrm{d}(\psi+\mathrm{i}\Delta\lambda)}\right|^2 \\
&= \left(\frac{m_0 A}{a}\right)^2 \left\{1+\left(\frac{1-n}{1+n}\tan\varphi\right)^2\right\} \left|\frac{\sigma-\mathrm{i}\tau}{\cosh(\psi+\mathrm{i}\Delta\lambda)}\right|^2 \\
&= \left(\frac{m_0 A}{a}\right)^2 \left\{1+\left(\frac{1-n}{1+n}\tan\varphi\right)^2\right\} \frac{\sigma^2+\tau^2}{\cosh^2\psi\cos^2\Delta\lambda+\sinh^2\psi\sin^2\Delta\lambda} \\
&= \left(\frac{m_0 A}{a}\right)^2 \left\{1+\left(\frac{1-n}{1+n}\tan\varphi\right)^2\right\} \frac{\sigma^2+\tau^2}{\sinh^2\psi+\cos^2\Delta\lambda} \\
&= \left(\frac{m_0 A}{a}\right)^2 \left\{1+\left(\frac{1-n}{1+n}\tan\varphi\right)^2\right\} \frac{\sigma^2+\tau^2}{\tan^2\chi+\cos^2\Delta\lambda},
\end{aligned}
\tag{36}
$$

5. Concluding remarks

A prospective explanation for the comprehensive derivation process of the extensive coordination conversion formulae, including those for meridian convergence and scale factor in the Gauss-Krüger projection has been presented.

It is hoped that this report will help to foster a better understanding of these useful formulae for all those who are concerned with geospatial information management.

References

Abramowitz, M. and Stegun, I. A., eds. (1964): Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover, New York, 74, http://people.math.sfu.ca/~cbm/aands/page_74.htm (accessed 11 Jan. 2012).

Deakin, R. E., Hunter, M. N. and Karney, C. F. F. (2011): The Gauss-Krueger Projection: Karney-Krueger equations, in Proc. XXV Intl. Cartographic Conf. (ICC2011), Paris, CO-300, http://icaci.org/documents/ICC_proceedings/ICC2011/Oral%20Presentations%20PDF/D1-Map%20 projection/CO-300.pdf (accessed 11 Jan. 2012).

Karney, C. F. F. (2011): Transverse Mercator with an accuracy of a few nanometers, Journal of Geodesy, 85(8), 475–485, doi: 10.1007/s00190-011-0445-3.

Kawase, K. (2011): A General Formula for Calculating Meridian Arc Length and its Application to Coordinate Conversion in the Gauss-Krüger Projection, Bulletin of the Geospatial Information Authority of Japan, 59, 1–13, http://www.gsi.go.jp/common/000062452.pdf (accessed 11 Jan. 2012).

Krüger, L. (1912): Konforme Abbildung des Erdellipsoids in der Ebene, Veröffentlichung Königlich Preuszischen geodätischen Institutes, Neue Folge, 52, Druck und Verlag von B. G. Teubner, Potsdam, 172p, doi: 10.2312/GFZ.b103-krueger28.

Weisstein, E. W. (2008): "Gudermannian." From MathWorld--A Wolfram Web Resource, http://mathworld.wolfram.com/Gudermannian.html (accessed 16 Jan. 2012).